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An Ordinal-Free Proof of the Cut-elimination Theorem for a Subsystem of Π_1^1-Analysis with ω-rule

Ryota Akiyoshi
Department of Philosophy
Keio University

概 要

The aim of this paper is to sketch our ideas of a simple ordinal-free proof of the cut-elimination theorem for a subsystem of Π_1^1-analysis with ω-rule.

The aim of this paper is to sketch our ideas of a simple ordinal-free proof of the cut-elimination theorem for a subsystem of Π_1^1-analysis with ω-rule.

The motivation is that use of heavy ordinal notation systems sometimes obscures our intuitive understanding of cut-elimination theorems. In the case of predicative systems, it is easy to understand why the cut-elimination procedure terminates. For example, the proof of the cut-elimination theorem for PA with ω-rule proceeds by induction on cut-degree. But the matter is not very transparent in the case of impredicative systems. Our proof of the cut-elimination theorem for a subsystem of Π_1^1-analysis with ω-rule proceeds just by transfinite induction on the height of a derivation. Moreover our proof involves only reasoning about well-founded trees.

The present paper consists of 5 sections. After recalling basic definitions in section 1, we introduce infintary systems BI_0^Ω, BI_1^Ω (section 2). BI_0^Ω is just cut-free arithmetic with ω-rule and Mints's "Repetition Rule". BI_1^Ω is obtained by adding cut-rule, a rule for second-order universal quantifier, and Buchholz's Ω, $\widetilde{\Omega}$-rules to BI_0^Ω. In section 3 we define operators $\mathcal{R}$, $\mathcal{E}$, and $\mathcal{E}_\omega$ on derivations in BI_1^Ω. Moreover we define the collapsing operator $\mathcal{D}_0$ which eliminates $\widetilde{\Omega}_{\neg\forall XA}$. Finally we define the substitution operator $\mathcal{S}_T^X$.

In section 4 we introduce BI_1^-, which is a subsystem of Π_1^1-analysis. BI_1 is obtained by adding $R_A, E, E_\omega, D_0, Sub_T^X$. These rules correspond to operations $\mathcal{R}$, $\mathcal{E}$, $\mathcal{E}_\omega$, D_0, and $\mathcal{S}_T^X$ respectively. The idea of introducing these devices is due to Buchholz[Buc91] to give a finite term rewriting system for continuous cut-elimination.

In section 5 we sketch our ideas of an ordinal-free proof of the cut-elimination theorem for BI_1. We define an embedding map g from derivations in BI_1 into the derivations in BI_1^Ω (5.1). Next we define for each derivation d in BI_1 functions $tp(d)$ and $d[i]$ (5.2). Finally we explain our ideas of an ordinal-free proof of the cut-elimination theorem for BI_1 (6.3). Our main observation is that $g(r(d))$ is a propei subderivation of $g(d)$ if $r(d)$ can be obtained from d by the proof-theoretic reduction for derivations in BI_1:

$$\begin{array}{ccc} \mathrm{BI}_1 : d & \xrightarrow{red} & r(d) \\ {\scriptstyle g}\downarrow & & {\scriptstyle g}\downarrow \\ \mathrm{BI}_1^\Omega : g^*(d) & \xrightarrow{>} & g^*(r(d)) \end{array}$$

where $g^*(d) > g^*(r(d))$ means that the height of $g^*(d)$ is strictly less than the height of $g^*(r(d))$. Therefore the cut-elimination theorem for BI_1 is proved by transfinite induction on $|d|$ (the height of d).

1 Preliminaries

First we define a language L which is the formal language of all systems considered below.

Definition 1 *Language L*

1. 0 is a term.
2. If t is a term, then $S(t)$ is a term.
3. If R is an n-ary predicate symbol for an n-ary primitive recursive relation, and $t_1,\ldots,t_n$ are terms, then $R(t_1,\ldots,t_n)$ is a formula. If X is unary predicate variable, and t is a term, then $X(t)$ is a formula. These formulas are called *atomic formulas.*
4. If A is an atomic formula, then $\neg A$ is a formula. A and $\neg A$ where A is atomic are called *literals.*
5. If A and B are formulas, then $A \wedge B$, $A \vee B$ are formulas.
6. If $A(0)$ is a formula, then $\forall x A(x)$, and $\exists x A(x)$ are formulas.
7. If A is formula, and A does contain no second order quantifier and no predicate variable except X, then $\forall X A$ and $\exists X A$ are formulas.

If A is a formula which is not atomic, then its *negation* $\neg A$ is defined using De Morgan's laws. The set of true literals is denoted as TRUE. T denotes an expression $\lambda x.A$ where $A(0)$ is a formula (called *abstraction*). Formulas which does not contain any second order quantifier are called *arithmetical.*

Remark 1 By the restriction, $A(X)$ is arithmetical if $\forall X A(X)$, or $\exists X A(X)$ is a formula.

Definition 2 $rk(A)$

1. $rk(A) := 0$ if A is a literal , $\forall X A(X)$, or $\exists X A(X)$.

2. $rk(A \wedge B) := rk(A \vee B) = sup(rk(A), rk(B)) + 1$.

3. $rk(\forall x A(x)) := rk(\exists x A(x)) = rk(A(0)) + 1$.

Remark 2 We remark that $rk(A) = 0$ if A is $\forall X A(X)$, or $\exists X A(X)$.

2 The Systems $\mathbf{BI}_0^\Omega$, $\mathbf{BI}_1^\Omega$

We define BI_0^Ω, BI_1^Ω using Buchholz's notation in [Buc01]. Only the *minor formulas* which occur in the premises of the rules, and the *principal formulas* which occur in the conclusions of the rules are explicitly shown. Any rule below is supposed to be closed under weakening, and contains contraction.

Let I be an inference symbol of a system. Then we write $\Delta(I)$, and $|I|$ in order to indicate the set of principal formulas of I, and the index set of I as in [Buc01], respectively. Moreover, $\bigcup_{i\in|I|}(\Delta_i(I))$ denotes the set of the minor formulas of I. If $d = I(d_i)_{|I|}$, then d_i denotes the subderivation of d indexed by i. If d is a derivation, $\Gamma(d)$ denotes its last sequent. Eigenvariables may occur free only in the premises, but not in the conclusions.

Definition 3 *The systems BI_0^Ω, BI_1^Ω*

The inference symbols of BI_0^Ω are

$$(\mathrm{Ax}_\Delta)\ \frac{}{\Delta} \text{ where } \Delta = \{A\} \subseteq \text{ TRUE or } \Delta = \{C, \neg C\}$$

$$(\bigwedge_{A_0 \wedge A_1})\ \frac{A_0 \quad A_1}{A_0 \wedge A_1} \qquad (\bigvee^k_{A_0 \vee A_1})\ \frac{A_k}{A_0 \vee A_1} \text{ where } k \in \{0, 1\}$$

$$(\bigwedge_{\forall x A})\ \frac{\ldots A(x/n) \ldots \text{ for all } n \in \omega}{\forall x A} \qquad (\bigvee^k_{\exists x A})\ \frac{A(x/k)}{\exists x A} \text{ where } k \in \omega$$

$$(Rep)\frac{\phi}{\phi}$$

The inference symbols of BI_1^Ω are obtained by adding the following inference symbols to those of BI_0^Ω.

$$(Cut_A)\ \frac{A \quad \neg A}{\phi} \qquad (\bigwedge^{Y}_{\forall XA})\ \frac{A(Y)}{\forall XA} \text{ where } Y \text{ is an eigenvariable}$$

$$(\Omega_{\neg\forall XA})\ \frac{\ldots\Delta_q^{\forall XA(X)}\ldots(q \in |\forall XA(X)|)}{\neg\forall XA}$$

$$(\tilde{\Omega}^{Y}_{\neg\forall XA})\ \frac{A(Y) \quad \ldots\Delta_q^{\forall XA(X)}\ldots(q \in |\forall XA(X)|)}{\phi} \text{ where } Y \text{ is an eigenvariable}$$

with

1. $\Delta_{(d,X)}^{\forall XA(X)} := \Gamma(d)\setminus \{A(X)\}$,
2. $\Gamma(d)$ is arithmetical,
3. $|\forall XA(X)| := \{(d, X) |\ d \in \mathrm{BI}_0^\Omega, X \notin FV(\Delta_{(d,X)}^{\forall XA(X)})\ \}$, and
4. $q = (d, X)$.

3 Cut-elimination Theorem for BI_1^Ω

Definition 4 $dg(I), dg(d)$

Let I be an inference symbol, and d be a derivation in BI_1. Then $dg(I)$, and $dg(d)$ are defined by

1. $dg(I) := rk(C) + 1$ if $I = Cut_C$.
2. $dg(I) := 0$ otherwise.
3. $dg(I(d_\tau)_{\tau\in|I|}) := sup(\{dg(I)\} \cup \{dg(d_\tau)|\tau \in |I|\})$.

We write $d \vdash_m \Gamma$ if $\Gamma(d) = \Gamma$, and $dg(d) \leq m$. Then we can prove the following theorems.

Theorem 1 *There exists an operator $\mathcal{R}_C$ on derivations in BI_1^Ω such that*

If $d_0 \vdash_m \Gamma, C$, $d_1 \vdash_m \Gamma, \neg C$, and $rk(C) \leq m$, then $\mathcal{R}_C(d_0, d_1) \vdash_m \Gamma$.

Theorem 2 *There is an operator $\mathcal{E}$ on derivations in BI_1^Ω such that*

If $d \vdash_{m+1} \Gamma$, then $\mathcal{E}(d) \vdash_m \Gamma$.

Theorem 3 *There is an operator $\mathcal{E}_\omega$ on derivations in BI_1^Ω such that*

If $d \vdash_\omega \Gamma$, then $\mathcal{E}_\omega(d) \vdash_0 \Gamma$.

Theorem 4 *There is an operator $\mathcal{D}_0$ on derivations in BI_1^Ω such that*

If $d \vdash_0 \Gamma$, and Γ is arithmetical, then $\mathrm{BI}_0^\Omega \ni \mathcal{D}_0(d) \vdash \Gamma$.

Corollary 1 *If $d \in BI_1^\Omega$ and $\Gamma(d)$ is arithmetical, then there exists d' such that $d' \in BI_0^\Omega$.*

Theorem 5 *There is an operator $\mathcal{S}$ such that*

If $\mathrm{BI}_0^\Omega \ni d \vdash \Gamma$, then $\mathrm{BI}_0^\Omega \ni \mathcal{S}_T^X(d) \vdash \Gamma[X/T]$.

4 The Systems $\mathbf{BI}_1^-$, $\mathbf{BI}_1$

We define BI_1^-, BI_1. Eigenvariables may occur free only in the premises, but not in the conclusions.

Definition 5 *The systems BI_1^-, BI_1*

The inference symbols of BI_1^- are

$$(\mathrm{Ax}_\Delta)\ \overline{\Delta} \text{ where } \Delta = \{A\} \subseteq \text{ TRUE or } \Delta = \{C, \neg C\}$$

$$(\bigwedge_{A_0 \wedge A_1})\ \frac{A_0 \quad A_1}{A_0 \wedge A_1} \qquad (\bigvee^k_{A_0 \vee A_1})\ \frac{A_k}{A_0 \vee A_1} \text{ where } k \in \{0,1\}$$

$$(\bigwedge_{\forall x A})\ \frac{\ldots A(x/n) \ldots \text{ for all } n \in \omega}{\forall x A} \qquad (\bigvee^k_{\exists x A})\ \frac{A(x/k)}{\exists x A} \text{ where } k \in \omega$$

$$(\bigwedge^Y_{\forall X A})\ \frac{A(Y)}{\forall X A} \text{ where } Y \text{ is an eigenvariable} \qquad (\bigvee^T_{\neg\forall X A})\ \frac{\neg A(X/T)}{\neg\forall X A}$$

$$(Cut_A)\frac{A, \neg A}{\phi}$$

The inference symbols of BI_1 are obtained by adding the following inference symbols to those of BI_1^-.

$$(R_A)\frac{C \quad \neg C}{\phi} \qquad (E)\frac{\phi}{\phi}$$

$$(E_\omega)\frac{\phi}{\phi} \qquad (D_0)\frac{\phi}{\phi}$$

$$(Sub_T^X)\frac{\Gamma}{\Gamma[X/T]}$$

Remark 3 These rules $E, E_\omega, D_0, Sub_T^X, R_C$ correspond to the operations $\mathcal{E}, \mathcal{E}_\omega, \mathcal{D}_0, \mathcal{S}_T^X, \mathcal{R}_C$ in the previous section.

5 Cut-elimination Theorem for BI_1

In this section, we sketch our idea of an ordinal-free proof of the cut-elimination theorem for BI_1 using one for BI_1^Ω.

We will define an embedding function g from derivations in BI_1 into the derivations in BI_1^Ω(5.1). Next we define functions $tp(d)$, $d[i]$ where d is a derivation in BI_1(5.2). Finally we explain our idea of an ordinal-free proof of the cut-elimination theorem for BI_1(5.3).

5.1 Interpretation of BI_1 in BI_1^Ω

Definition 6 *Embedding fuction g*

Let d be a derivation in BI_1. Then we define the function g by induction on d as follows.

1. $g(\mathrm{Ax}_\Delta) := \mathrm{Ax}_\Delta$.
2. $g(\bigwedge_{A_0 \wedge A_1}(d_0, d_1)) := \bigwedge_{A_0 \wedge A_1}(g(d_0), g(d_1))$.
3. $g(\bigvee^k_{A_0 \vee A_1}(d_0)) := \bigvee^k_{A_0 \vee A_1}(g(d_0))$.
4. $g(\bigwedge_{\forall x A}(d_n)_{n\in\omega}) := \bigwedge_{\forall x A}(g(d_n))_{n\in\omega})$.
5. $g(\bigvee^k_{\exists x A}(d_0)) := \bigvee^k_{\exists x A}(g(d_0))$.
6. $g(\bigwedge_{\forall X A}(d_0)) := \bigwedge_{\forall X A}(g(d_0))$.
7. $g(\bigvee^T_{\neg\forall X A}(d_0)) := \Omega(\mathcal{R}_{A(T)}(\mathcal{S}_T^X(d_q), g(d_0)))_{q\in|\forall X A(X)|}$ where $(d_q, X) = q \in |\forall X A(X)|$.

8. $g(Cut_C(d_0, d_1)) := Cut_C(g(d_0), g(d_1))$.

9. $g(E(d_0)) := \mathcal{E}(g(d_0))$.

10. $g(E_\omega(d_0)) := \mathcal{E}_\omega(g(d_0))$.

11. $g(D_0(d_0)) :=$

    (a) $\mathcal{D}_0(g(d_0))$ if $g(d_0)$ satisfies the conditions in the collapsing theorem.

    (b) $g(d_0)$ otherwise.

12. $g(Sub_T^X(d_0)) :=$

    (a) $\mathcal{S}_T^X(g(d_0))$ if $g(d_0)$ satisfies the conditions in the substitution theorem.

    (b) $g(d_0)$ otherwise.

13. $g(R_C(d_0, d_1)) := \mathcal{R}_C(g(d_0), g(d_1))$.

Remark 4

1. Let $d = \bigvee^T_{\neg\forall X A(X)}(d_0)$. Then $g(d)$ is the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Delta_q, A(X)}}{\Delta_q, A(T)}\, \mathcal{S}_T^X \qquad \dfrac{\vdots}{\Gamma, \neg A(T), \neg\forall X A(X)}}{\ldots\Gamma, \Delta_q, \neg\forall X A(X)\ldots}\, \mathcal{R}_{A(T)}}{\Gamma, \neg\forall X A(X)}\, \Omega$$

2. g replaces rules E, E_ω, D_0, Sub_T^X, R_C by the corresponding operations $\mathcal{E}$, $\mathcal{E}_\omega$, $\mathcal{D}_0$, $\mathcal{S}_T^X$, $\mathcal{R}_C$ respectively. But it preserves Cut_C : $g(Cut_C(d_0, d_1)) = Cut_C(g(d_0), g(d_1))$.

Definition 7 $dg(d)$

Let d be a derivation in BI_1. Then $dg(d)$ is defined by

1. $dg(d) := max(rk(A(T)), dg(d_0))$ if $I = \bigvee^T_{\neg\forall X A(X)}$.

2. $dg(d) := max(rk(C) + 1, dg(d_0), dg(d_1))$ if $I = Cut_C$.

3. $dg(d) := dg(d_0) \dot{-} 1$ if $I = E$.

4. $dg(d) := 0$ if $I = E_\omega$.

5. $dg(d) := max(rk(C), dg(d_0), dg(d_1))$ If $I = R_C$.

6. $dg(I(d_\tau)_{\tau \in |I|}) := sup\{dg(d_\tau) | \tau \in |I|\}$ otherwise.

We write $d \vdash_m \Gamma$ if $\Gamma(d) = \Gamma$, and $dg(d) \leq m$. Next we define the notion of *proper derivations* such that the operations $\mathcal{D}_n$, and $\mathcal{S}_T^X$ have to be applied to only subderivations satisfying the conditions in Theorems 4, 5 respectively.

Definition 8 *A derivation d in BI_1 is called proper if*

1. for each subderivation $D_0(h_0)$ of d, $dg(h_0) = 0$, and $\Gamma(h_0)$ is arithmetical,

2. for each subderivation $Sub_T^X(h)$ of d, h is of the form $D_0(h_0)$.

Theorem 6 *Let d be a proper derivation of Γ in BI_1. Then $g(d) \vdash_{dg(d)} \Gamma$.*

5.2 Definition of $tp(d)$, and $d[i]$

Now we can define $tp(d)$, and $d[i]$ where $i \in |tp(d)|^*$ for each proper derivation $d \in \mathrm{BI}_1$ such that

1. $tp(d)$ is the last inference symbol of $g(d)$.

2. $d[i]$ is also a proper derivation in BI_1.

3. $g(d[i])$ is the i-th immediate subderivation of $g(d)$.

In fact the situation is more complicated because for d with $tp(d) = \Omega$ or $\widetilde{\Omega}$ elements of the index set may be themselves derivations.

Definition 9 $|\forall X A|^*, |I|^*, g(q)$

We define $|\forall X A|^*$, $|I|^*$ where I is an inference symbol of BI_1^Ω and $g(q)$ where $q = (d, X) \in |\forall X A|^*$ as follows:

1. $|\forall X A|^* := \{(d, X) |$ d is of the form $D_0(d')$ where d is a proper derivation in $\mathrm{BI}_1, X \notin FV(\Delta_{(d,X)}^{\forall X A(X)})\}$ with

   (a) $\Delta_{(d,X)}^{\forall X A(X)} = \Gamma(d) \setminus \{A(X)\}$, and

   (b) $\Delta_{(d,X)}^{\forall X A(X)}$ is arithmetical.

2. $|\Omega_{\neg \forall X}|^* := |\forall X A|^*$.

3. $|\widetilde{\Omega}_{\neg \forall X}^X|^* := \{0\} \cup |\forall X A|^*$.

4. $|I|^* := |I|$ if $I \neq \Omega_{\neg\forall X}$ or $\widetilde{\Omega}^X_{\neg\forall X}$.

5. $g(q) := (g(d), X)$ where $q = (d, X) \in |\forall XA|^*$.

Definition 10 *tp*$(d), d[i]$

By primitive recursion on d, we define $tp(d) \in \mathrm{BI}_1^\Omega$, and derivations $d[i]$ where $i \in |\mathrm{tp}(d)|^*$. We assume that *separation of eigenvariables*: all eigenvariables in d are distinct and none of them occurs below the inference in which it is used as an eigenvariable.

1. $d = \mathrm{Ax}_\Delta : tp(d) := \mathrm{Ax}_\Delta$.
2. $d = \bigwedge_{A_0 \wedge A_1}(d_0, d_1) : tp(d) := \bigwedge_{A_0 \wedge A_1}, d[i] := d_i$.
3. $d = \bigvee^k_{A_0 \vee A_1}(d_0) : tp(d) := \bigvee^k_{A_0 \vee A_1}, d[0] := d_0$.
4. $d = \bigwedge_{\forall x A}(d_i)_{i \in \omega} : tp(d) := \bigwedge_{\forall x A}, d[i] := d_i$.
5. $d = \bigvee^k_{\exists x A}(d_0) : tp(d) := \bigvee^k_{\exists x A}, d[0] := d_0$.
6. $d = \bigwedge_{\forall X A}(d_0) : tp(d) := \bigwedge_{\forall X A}, d[0] := d_0$.
7. $d = \bigvee^T_{\neg\forall X A(X)}(d_0) : tp(d) := \Omega_{\neg\forall XA}, d[(h, X)] := R_{A(T)}(Sub^X_T(h), d_0)$.
8. $d = Cut_A(d_0, d_1) : tp(d) := Cut_A, d[i] := d_i$.
9. $d = E(d_0)$:
   (a) $\mathrm{tp}(d_0) = Cut_C : tp(d) := Rep, d[0] := R_C(E(d_0[0]), E(d_0[1]))$.
   (b) otherwise: $\mathrm{tp}(d) = tp(d_0), d[i] := E(d_0[i])$.
10. $d = E_\omega(d_0)$:
    (a) $tp(d_0) = Cut_C : tp(d) := Rep, d[0] := E^{n+1}(Cut_C(E_\omega(d_0[0]), E_\omega(d_0[1])))$ where $rk(C) = n$, and E^{n+1} denotes $n+1$-times applications of E-rule.
    (b) otherwise: $tp(d) := tp(d_0), d[i] := E_\omega(d_0[i])$.
11. $d = D_0(d_0)$:
    (a) $tp(d_0) = \widetilde{\Omega}^Y : tp(d) := Rep, d[0] := D_0(d_0[(D_0(d_0[0]), Y)])$.
    (b) otherwise: $tp(d) := tp(d_0), d[i] := D_0(d_0[i])$.
12. $d = Sub^X_T(d_0) : tp(d) := tp(d_0)[X/T], d[i] := Sub^X_T(d_0[i])$.
13. $d = R_A(d_0, d_1)$:

(a) $A \notin \Delta(tp(d_0)) : tp(d) := tp(d_0), d[i] := R_A(d_0[i], d_1)$.

(b) $\neg A \notin \Delta(tp(d_1)) : tp(d) := tp(d_1), d[i] := R_A(d_0, d_1[i])$.

(c) $A \in \Delta(tp(d_0))$, and $\neg A \in \Delta(tp(d_1))$:

i. $tp(d_0) = \mathrm{Ax}_\Delta : tp(d) := Rep$, and $d[0] := d_1$.

ii. $tp(d_1) = \mathrm{Ax}_\Delta : tp(d) := Rep$, and $d[0] := d_0$.

iii. $A = A_0 \wedge A_1 : tp(d_0) = \bigwedge_{A_0 \wedge A_1}$, and $tp(d_1) = \bigvee^k_{\neg A_0 \vee \neg A_1}$ for some $k \in \{0, 1\}$. $tp(d) := Cut_{A_k}, d[0] := R_A(d_0[k], d_1), d[1] := R_A(d_0, d_1[0])$.

iv. $A = A_0 \vee A_1$, $\forall x A$, or $\exists x A$: similarly to the case of $A_0 \wedge A_1$.

v. $A = \forall X A : tp(d_0) = \bigwedge^Y_{\forall X A}$, and $tp(d_1) = \Omega_{\neg \forall X A}$. $tp(d) := \tilde{\Omega}^Y_{\neg \forall X A}, d[0] := R_{\forall X A}(d_0[0], d_1), d[q] := R_{\forall X A}(d_0, d_1[q])$ for $q \in |\forall X A|^*$.

vi. $A = \exists X A$: similarly to the case of $\forall X A$.

Theorem 7 *Assume that* $BI_1 \ni d \vdash_m \Gamma$ *is a proper derivation, and* $i \in |tp(d)|^*$. *Then the following properties hold:*

1. $d[i]$ is also a proper derivation in BI_1.

2. $d[i] \vdash_m \Gamma, \Delta_i(tp(d))$.

3. $dg(d[i]) \leq dg(d)$.

4. If $tp(d) = Cut_A$, then $rk(A) < dg(d)$.

5.3 Cut-elimination Theorem for $\mathbf{BI}_1$

In this section, we explain our ideas of the cut-elimination theorem for BI_1. Let *red* be a suitable reduction relation between derivations in BI_1. Instead of defining *red* explicitly, we explain it using examples. Define $|I(d_i)_{i \in |I|}| := sup(|d_i| + 1)_{i \in |I|}$. Then $|d| < |d'|$ if d is a proper subderivation d'.

Lemma 1 *Assume that* $d = E(Cut_C(d_0, d_1))$, *and* $r(d) = R_C(E(d_0), E(d_1))$. *Then* $|g(d)| > |g(r(d))|$.

Proof. $g(r(d)) = \mathcal{R}_C(\mathcal{E}(g(d_0)), \mathcal{E}(g(d_1)))$. On the other hand $g(d) = g(E(Cut_C(d_0, d_1))) = \mathcal{E}(Cut_C(g(d_0), g(d_1))) = Rep(R_C(\mathcal{E}(g(d_0)), \mathcal{E}(g(d_1))))$ (note that g preserves Cut_C). Therefore $|g(d)| > |g(r(d))|$. □

Next we see $|g(d)| > |g(r(d))|$ in the case of axiom-reduction.

Lemma 2 *Assume that $d = R_C(d_0, d_1)$, d_0 is an axiom $C, \neg C$, and $r(d) = d_1$. Then $|g(d)| > |g(r(d))|$.*

Proof.
$g(R_C(d_0, d_1)) = \mathcal{R}_C(g(d_0), g(d_1))) = \mathcal{R}_C(\mathrm{Ax}_{C,\neg C}, g(d_1)) = Rep(g(d_1))$. Therefore $|g(d)| > |g(r(d))|$. □

Lemma 3 *Assume that $d = E(R_{C_0 \wedge C_1}(\bigwedge_{C_0 \wedge C_1}(d_{000}, d_{001}), \bigvee^k_{\neg C_0 \vee \neg C_1}(d_{010})))$, and $r(d) = R_{C_k}(E(R_C(d_{00k}, d_{01})), E(R_C(d_{00}, d_{010})))$. Then $|g(d)| > |g(r(d))|$.*

Proof.

$$
\begin{aligned}
& g(E(R_C(\bigwedge\nolimits_{C_0 \wedge C_1}(d_{000}, d_{001}), \bigvee\nolimits^k_{\neg C_0 \vee \neg C_1}(d_{010})))) \\
& = \mathcal{E}(\mathcal{R}_C(\bigwedge\nolimits_{C_0 \wedge C_1}(g(d_{000}), g(d_{001})), \bigvee\nolimits^k_{\neg C_0 \vee \neg C_1}(g(d_{010})))) \\
& = \mathcal{E}(Cut_{C_k}(\mathcal{R}_C(g(d_{00k}), g(d_{01})), \mathcal{R}_C(g(d_{00}), g(d_{010})))) \\
& = Rep(\mathcal{R}_{C_k}(\mathcal{E}(\mathcal{R}_C(g(d_{00k}), g(d_{01}))), \mathcal{E}(\mathcal{R}_C(g(d_{00}), g(d_{010}))))).
\end{aligned}
$$

On the other hand, $g(r(d)) = \mathcal{R}_{C_k}(\mathcal{E}(\mathcal{R}_C(g(d_{00k}), g(d_{01}))), \mathcal{E}(\mathcal{R}_C(g(d_{00}), g(d_{010}))))$. Therefore $|g(d)| > |g(r(d))|$. □

Lemma 4 *Assume that $d = E^{m+1}(R_C(\bigwedge_{\forall X C_0(X)}(d_{000}), \bigvee^T_{\exists X \neg C_0(X)}(d_{010})))$, and $E^{m+1}(R_C(\bigwedge_{\forall X C_0(X)}(d_{000}), R_{C_0(T)}(Sub^X_T(d_{01q}), g(d_{010}))))$. Then $|g(d)| > |g(r(d))|$.*

Proof.
According to the definition of g,

$$
\begin{aligned}
& g(E^{m+1}(R_C(\bigwedge\nolimits_{\forall X C_0(X)}(d_{000}), \bigvee\nolimits^T_{\exists X \neg C_0(X)}(d_{010})))) \\
& = \mathcal{E}^{m+1}(\mathcal{R}_C(\bigwedge\nolimits_{\forall X C_0(X)}(g(d_{000})), \Omega(\mathcal{R}_{C_0(T)}(S^X_T(d_{01q}), g(d_{010}))_{q \in |\forall X A(X)|}))) \\
& = \widetilde{\Omega}(\mathcal{E}^{m+1}(\mathcal{R}_C(g(d_{000}), g(d_{01}))), \mathcal{E}^{m+1}(\mathcal{R}_C(\bigwedge\nolimits_{\forall X C_0(X)}(g(d_{000})), \mathcal{R}_{C_0(T)}(S^X_T(d_{01q}), g(d_{010})))))_q.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
& g(r(d)) \\
& = \mathcal{E}^{m+1}(\mathcal{R}_C(\bigwedge\nolimits_{\forall X C_0(X)}(g(d_{000})), \mathcal{R}_{C_0(T)}(S^X_T(\mathcal{D}_0(\mathcal{E}^{m+1}(\mathcal{R}_C(g(d_{000}), g(d_{01})))), g(d_{010}))))).
\end{aligned}
$$

with $\mathcal{D}_0(\mathcal{E}^{m+1}(\mathcal{R}_C(g(d_{000}), g(d_{01})))) \in |\forall X C_0(X)|$. Therefore $|g(d)| > |g(r(d))|$. □

Remark 5 Using Ω or $\widetilde{\Omega}$-rule, we can list up *all* possible cuts in the cut-elimination process. Lemma 4 shows that the result of Takeuti's reduction is one of such cuts.

From these lemmas, we can see the following diagram in the essential reductions which we have considered:

$$\begin{array}{ccc} d & \xrightarrow{red} & r(d) \\ {\scriptstyle g}\downarrow & & {\scriptstyle g}\downarrow \\ g(d) & \xrightarrow{>} & g(r(d)) \end{array}$$

where $g(r(d))$ is a subderivation of $g(d)$. A derivation d in BI_1 is *cut-free* if d does not contain Cut_A, R_A. Therefore we can prove the cut-elimination theorem for BI_1 by transfinite induction on the height of $g(d)$.

Theorem 8 *Let d be a proper derivation of Γ in BI_1 such that Γ is arithmetical, and $dg(d) = 0$. Then there exists a cut-free derivation d' of the same sequent Γ.*

Corollary 2 *Let d be a proper derivation of Γ in BI_1 such that Γ is arithmetical. Then there exists a cut-free derivation d' of the same sequent Γ.*

A derivation d in BI_1^- is *cut-free* if d does not contain Cut_A. Then we can prove the following corollary.

Corollary 3 *Let d be a derivation of Γ in BI_1^- such that Γ is arithmetical. Then there exists a cut-free derivation d' in BI_1^- of the same sequent Γ.*

Remark 6 The full version of this paper is [Aki08]. Our proof can be extended into the full Π^1_1-CA [AM08].

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